# Lecture 10
# Link Analysis III / Frequent Itemsets I

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# TODAY

*Overview*

- *Link Analysis III*
  - Hubs and Authorities: Alternative, Non-PageRank Approach
- *Frequent Itemsets I*
  - The Market-Basket Model
  - Frequent Itemsets: Definition and Applications
  - Association Rules
  - The A-Priori Algorithm

*Learning Goals:* Understand these topics and get familiarized

*Hubs and Authorities*

# Hubs and Authorities: Introduction

- The hubs-and-authorities algorithm, also called *HITS (hyperlink-induced topic search)*, is an alternative to PageRank
- *Similarities:*
  - Quantifies importance of pages
  - Involves fixedpoint computation by iterative matrix-vector multiplication
- *Differences:*
  - Divides pages into hubs and authorities
  - Not a preprocessing step: ranks importance of responses to query

# HITS: INTUITION

- Importance is twofold
- *Authorities* are pages deemed to be valuable because they provide information on a topic
  - E.g. course website at university
- *Hubs* are pages deemed to be valuable because of providing directions about topics
  - E.g. department directory providing links to all course websites
- Mutually recursive definition:
  - *Good hub* links to good authorities
  - *Good authority* is linked to by good hubs

# Hubbiness and Authority: Definition

## Definition [Hubbiness, Authority]

- Let the number of webpages be $n$
- Let $\mathbf{h} \in \mathbb{R}^n, \mathbf{a} \in \mathbb{R}^n$ be two vectors where
  - $\mathbf{h}_i$ quantifies the goodness of page $i$ as a hub
  - $\mathbf{a}_i$ quantifies the goodness of page $i$ as an authority
- $\mathbf{h}_i$ is also referred to as *hubbiness* of page $i$

## Remark

- Values of $\mathbf{h}, \mathbf{a}$ are generally scaled such that
  - *either* the largest component is 1
  - *or* the sum of components is 1
  - In the following, first option will be used here

# Link Matrix: Definition

## Definition [Link Matrix]

- Let the number of webpages be $n$
- The *link matrix* $L \in \{0,1\}^{n \times n}$ of the Web is defined by

$$L_{ij} = \begin{cases} 1 & \text{there is a link from page } i \text{ to page } j \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

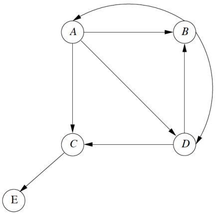
- Its transpose $L^T$ is defined by $L^T_{ij} = L_{ji}$, that is $L^T_{ij} = 1$ if there is a link from the $j$-th to the $i$-th page, and zero otherwise

## Remark

- $L^T$ is similar to the PageRank web transition matrix $M$ insofar as

$$L^T_{ij} \neq 0 \quad \text{if and only if} \quad M_{ij} \neq 0$$

# Link Matrix: Example




Example web graph

Adopted from `mmds.org`

$$L = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad L^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Corresponding link matrix and its transpose

Adopted from `mmds.org`

# Hubs and Authorities: Formal Relationship

- Good hub links to good authorities:

$$\mathbf{h}_i = \lambda \sum_{j=1}^{n} L_{ij}\mathbf{a}_j \qquad \text{or, equivalently} \qquad \mathbf{h} = \lambda L\mathbf{a} \tag{2}$$

  where $\lambda$ represents the necessary scaling of $\mathbf{h}$

- Good authority is linked to by good hubs:

$$\mathbf{a}_i = \mu \sum_{j=1}^{n} L_{ij}^{T}\mathbf{h}_j \qquad \text{or, equivalently} \qquad \mathbf{a} = \mu L^{T}\mathbf{h} \tag{3}$$

  where $\mu$ represents the necessary scaling of $\mathbf{a}$.

# Hubs and Authorities: Formal Relationship

- Substituting (3) into (2) yields:

$$\mathbf{h} = \lambda\mu L L^T \mathbf{h} \tag{4}$$

- Substituting (2) into (3) yields:

$$\mathbf{a} = \mu\lambda L^T L \mathbf{a} \tag{5}$$

- $\mathbf{h}, \mathbf{a}$ can be determined by solving linear equations
- *However:* $LL^T, L^T L$ are not sufficiently sparse for their size to allow for solving corresponding linear equations
- *Solution:* HITS algorithm

# The HITS Algorithm

*Initialization:* Set $\mathbf{h}_i = 1$ for all $i$, that is $\mathbf{h} = (1, ..., 1)$

*Iteration:*

1. Compute
$$\mathbf{a} = L^T\mathbf{h}$$
2. Scale such that largest component of **a** is 1
3. Compute
$$\mathbf{h} = L\mathbf{a}$$
4. Scale such that largest component of **h** is 1
5. Repeat until convergence

# HITS Algorithm: Example

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 3/2 \\ 1/2 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 2/3 \\ 0 \end{bmatrix}$$

$$\mathbf{h} \quad L^{\mathrm{T}}\mathbf{h} \quad \mathbf{a} \quad L\mathbf{a} \quad \mathbf{h}$$

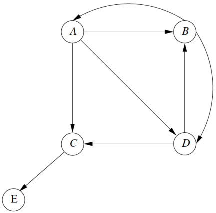
$$\begin{bmatrix} 1/2 \\ 5/3 \\ 5/3 \\ 3/2 \\ 1/6 \end{bmatrix} \quad \begin{bmatrix} 3/10 \\ 1 \\ 1 \\ 9/10 \\ 1/10 \end{bmatrix} \quad \begin{bmatrix} 29/10 \\ 6/5 \\ 1/10 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 12/29 \\ 1/29 \\ 20/29 \\ 0 \end{bmatrix}$$

$$L^{\mathrm{T}}\mathbf{h} \quad \mathbf{a} \quad L\mathbf{a} \quad \mathbf{h}$$

First two iterations of HITS algorithm

Adopted from `mmds.org`

# HITS ALGORITHM: EXAMPLE




A and D are good hubs, B and C are good authorities

Adopted from mmds.org

$$\mathbf{h} = \begin{bmatrix} 1 \\ 0.3583 \\ 0 \\ 0.7165 \\ 0 \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} 0.2087 \\ 1 \\ 1 \\ 0.7913 \\ 0 \end{bmatrix}$$

Limits of **h**, **a** on graph

Adopted from mmds.org

*Frequent Itemsets*
*Introduction*

# Frequent Itemsets: Overview

*Foundations*

- There are *items* available in the market
- There are *baskets*, sets of items having been purchased together
- A *frequent itemset* is a set of items that is found to commonly appear in many baskets
- The *frequent-itemset problem* is to identify frequent itemsets

# Market-Basket Model

*Market-basket model*

- The market-basket model is a *many-many-relationship*
  - One basket holds many items
  - One item appears in several baskets
- Each basket is an itemset, i.e. a set of (one or several) items
- Usually, the number of items in a basket is small compared to number of items overall
- Number of baskets is usually large; too large to fit in main memory
- Data usually is a sequence of baskets

# Frequent Itemsets: Definition

Definition [Frequent Itemset]:

- Let $s > 0$ be a *support threshold*
- Let $I$ be a set of items
- $\text{supp}(I)$, the *support* of $I$, is the number of baskets in which $I$ appears as a subset

An itemset $I$ is referred to as *frequent* if

$$\text{supp}(I) \geq s \tag{6}$$

that is, if the support of $I$ is at least the support threshold

# FREQUENT ITEMSETS: EXAMPLE

*Baskets*

1. {and, dog, bites}
2. {news, claims, a, cat, mated, with, a, dog, and, produced, viable, offspring}
3. {cat, killer, likely, is, a, big, dog}
4. {professional, free, advice, on, dog, training, puppy, training}
5. {cat, and, kitten, training, behavior}
6. {dog, cat, provides, training, in, Oregon}
7. {dog, and, cat, is, a, slang, term, used, by, police, officers, for, a, male-female, relationship}
8. {shop, for, your, show, dog, grooming, and, pet, supplies}

- E.g. supp({dog}) = 7, supp({and}) = 5, supp({dog, and}) = 4
- Let the support threshold $s = 3$
- 5 frequent singletons: {dog},{cat},{a},{and},{training}
- 5 frequent doubletons: {dog, a},{dog, and},{dog, cat},{cat, a},{cat, and}
- 1 frequent triple: {dog, cat, a}

# Frequent Itemsets: Applications

- *Retailers / Supermarkets / Chain stores*
  - *Items:* Products offered
  - *Baskets:* Sets of products purchased by one customer during one shopping run
  - *Frequent Itemsets:* Products purchased together unusually often
    ☞ Beer and diapers
- *Related concepts*
  - *Items:* Words, excluding stop words
  - *Baskets:* News articles, documents
  - *Frequent Itemsets:* Groups of words representing joint concept
- *Plagiarism*
  - *Items:* Documents
  - *Baskets:* Sentences
  - *Frequent Itemsets:* Documents containing unusually many sentences in common

# Association Rules

- Let $j$ be an item and $I$ be an itemset
- An association rule
$$I \rightarrow j$$
expresses that if $I$ is likely to appear in a basket, so is $j$
- In other words, if $I$ shows in basket, one is confident to assume that $j$ does, too

Definition [Confidence]:
The *confidence* of a rule $I \rightarrow j$ is defined as

$$\frac{\text{supp}(I \cup \{j\})}{\text{supp}(I)} \tag{7}$$

that is the fraction of baskets containing $I$, that also contain $j$.

# Association Rules: Confidence

Definition [Confidence]:
The *confidence* of a rule $I \to j$ is defined as

$$\frac{\text{supp}(I \ \cup \ \{j\})}{\text{supp}(I)}$$

that is the fraction of baskets containing $I$, that also contain $j$.

*Example from above*

- Confidence of $\{cat, dog\} \to and$ is 3/5
- Confidence of $\{cat\} \to kitten$ is 1/6

# Association Rules: Interest

- Let $n$ be the number of baskets overall
- Confidence for $I \to j$ can be meaningless if fraction of baskets containing $j$ is large
- Confidence may just reflect that fraction
- So presence of $I$ does not increase confidence to see $j$ as well
- *Interest* is supposed to put this into context

Definition [Interest]:
The *interest* of a rule $I \to j$ is defined as

$$\frac{\text{supp}(I \cup \{j\})}{\text{supp}(I)} - \frac{\text{supp}(\{j\})}{n} \tag{8}$$

that is the confidence of $I \to j$ minus the fraction of baskets that contain $j$

# Association Rules: Interest

Definition [Interest]:
The *interest* of a rule $I \rightarrow j$ is defined as

$$\frac{\text{supp}(I \cup \{j\})}{\text{supp}(I)} - \frac{\text{supp}(\{j\})}{n}$$

that is the confidence of $I \rightarrow j$ minus the fraction of baskets that contain $j$

*Examples*

- $\{diapers\} \rightarrow beer$ was found to have great interest
- $\{dog\} \rightarrow cat$ has interest $5/7 - 3/4 = -0.036$
- $\{cat\} \rightarrow kitten$ has interest $1/6 - 1/8 = 0.042$

# Frequent Itemsets to Association Rules

*Situation*

- Consider frequent itemsets of "reasonably high" support $s$
  - Note that each frequent itemset suggests to be acted upon
    ☞ keep their number reasonably low
  - Reasonably low often means about 1% of baskets
- Confidence for a rule $I \to j$ should be at least (about) 50%
  ☞ Support for $I \cup \{j\}$ also fairly high

*Procedure*

- Assume all $I$ with $\text{supp}(I) \geq s$ have been mined
- For $J$ of $n$ items with $\text{supp}(J) \geq s$, there are $n$ possible association rules $J \setminus \{j\} \to J$ (where each $j$ is one of the $n$ items)
- $\text{supp}(J) \geq s$ implies $\text{supp}(J \setminus \{j\}) \geq s$
- Confidence of $J \setminus \{j\} \to J$ is easily computed as

$$\frac{\text{supp}(J)}{\text{supp}(J \setminus \{j\})}$$

*Mining Frequent Itemsets*
*The A-Priori Algorithm*

# Market-Basket Data: Representation

- Market-basket data is stored in a file basket-by-basket
  - If items refer to identifiers, for example $\{3, 36, 99\}\{6, 78, 11\}\ldots$
- *Assumption:* Average size of basket is rather small
- *Usually*, file does not fit in main memory
- Generating all subsets of size $k$ for a basket of size $n$ requires

$$\binom{n}{k} \approx \frac{n^k}{k!}$$

runtime

# Market-Basket Data: Representation

- Generating all subsets of size $k$ for a basket of size $n$ requires

$$\binom{n}{k} \approx \frac{n^k}{k!}$$

  runtime

- *This often is little time because:*
- $n$ was assumed to be small
- $k$ is usually very small
- When $k$ is large, one can virtually reduce $n$ further by removing infrequent items

# Market-Basket Data: Runtime Consideration

*Insight*

- Runtime dominated by taking data from disk to main memory
- *Consequence:* Processing all baskets is proportional to size of file
- *Runtime* proportional to number of passes through file
- For a *fast frequent itemset mining* algorithm:

**Limit number of passes through basket file**

# Use of Main Memory

- *Issue:* One needs to store counts for itemsets of size $k$
  - There could be many such itemsets
  - How to store these counts?
- *Consequence:* There is a limit on the number of items an algorithm can deal with
- *Example:*
  - Let there be $n$ items
  - For counting pairs, we need to store $\binom{n}{2} \approx n^2/2$ counts
  - Integers of 4 bytes: need $2n^2$ bytes to store counts
  - Consider machine of 2 GB, or $\approx 2^{31}$ bytes of main memory
  - Then $n < 2^{15} \approx 33\,000$ is required
- *Note:* Items can be hashed to integers, if they are not integers

# Storing Itemset Counts: The Triangular-Matrix Method

- In the following, consider storing itemsets of size 2
  - Remember that support threshold is quite large in real applications
  - So, many more pairs than triples, quadruples and so on in real applications
- *Insight:* Storing counts $a[i,j]$ in matrix $A = (a[i,j])_{1 \leq i < j \leq n} \in \mathbb{N}^{n \times n}$ wastes half of $A$

# Storing Itemset Counts: The Triangular-Matrix Method

- *Insight:* Storing counts $a[i,j]$ in matrix $A = (a[i,j])_{1 \leq i < j \leq n} \in \mathbb{N}^{n \times n}$ wastes half of $A$
- *Solution:* Store count for pair of items $\{i,j\}, 1 \leq i < j \leq n$ in

$$a[k] \quad \text{where} \quad k = (i-1)(n - \frac{i}{2}) + j - i \tag{9}$$

  This stores pairs in lexicographical order

$$\{1,2\},\{1,3\},...,\{1,n\},\{2,3\},...,\{2,n\},...,\{n-2,n\},\{n-1,n\}$$

# Storing Itemset Counts: The Triples Method

- Store triples $[i, j, c]$ for all pairs $\{i, j\}$ whose count $c > 0$
- For example, do this with hash table, hashing $i, j$ as search key
- *Advantage:* Does not require space for pairs $\{i, j\}$ of count zero
- *Disadavantage:* Requires three times the space if $c > 0$
- *Rationale:* Triangular matrix method better if at least 1/3 of the $\binom{n}{2}$ pairs appear in basket

# Storing Itemset Counts: Example

*Example*

- Consider
  - 100 000 items
  - 10 000 000 baskets of
  - 10 items each
- Triangular-matrix method: $\binom{10^5}{2} \approx 5 \times 10^9$ integer counts
- Triples method: $10^7 \binom{10}{2} \approx 4.5 \times 10^8$ counts, making for $3 \times 4.5 \times 10^8 = 1.35 \times 10^9$ integers to be stored
- Triples method proves to be more appropriate

# Monotonicity

Theorem [Monotonicity]:

- Let $s$ be the support threshold.
- Let $I, J$ be sets such that $J \subseteq I$

Then if $I$ is frequent, any subset $J$ of $I$ is, too:

$$\text{supp}(I) \geq s \quad \text{implies} \quad \text{supp}(J) \geq s \tag{10}$$

Proof.
Each basket that holds $I$ also holds $J$, as $J$ is contained in $I$. So, the number of baskets that hold $J$ is at least as large as the number of baskets that hold $I$. □

# Maximal Frequent Itemset I

Definition [Maximal Frequent Itemset]:

- Let $s$ be the support threshold.
- Let $I$ be frequent, that is $\text{supp}(I) \geq s$.

$I$ is said to be *maximal* if no superset of $I$ is frequent:

$$\text{for all } J \supsetneq I : \text{supp}(J) < s \tag{11}$$

# MAXIMAL FREQUENT ITEMSET II

DEFINITION [MAXIMAL FREQUENT ITEMSET]:
$I$ is said to be *maximal* if no superset of $I$ is frequent:

$$\text{for all } J \supsetneq I : \text{supp}(J) < s \tag{12}$$

*Example (from above):*

- At support threshold $s = 3$, we found frequent pairs $\{dog, a\}, \{dog, and\}, \{dog, cat\}, \{cat, a\}, \{cat, and\}$
- $\{dog, cat, a\}$ was found the only frequent triple
- ☞ $\{dog, cat, a\}, \{dog, and\}$ and $\{cat, and\}$ are maximal, while $\{dog, a\}, \{dog, cat\}, \{cat, a\}$ are not

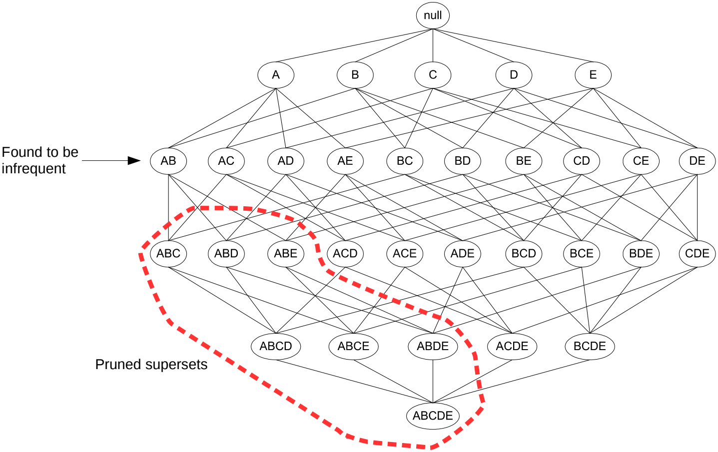
# Note on Counting Pairs I

- The number of frequent pairs is larger than frequent triples, quadruples, ... why?
- For making sense, number of (maximal) frequent itemsets is supposed to be sufficiently small
  - Human applicants need to work it out on all of them
- So, support threshold is set sufficiently high

# Note on Counting Pairs II

- Any maximal frequent itemset holds many more smaller, non-maximal frequent itemsets
- The resulting situation implies that there are many more frequent pairs than triples, many more frequent triples than quadruples, and so on
- *Important:*
  - Still, the possible number of triples, quadruples is (much) greater than pairs
  - Any good frequent itemset *algorithm needs to avoid running through all possible triples, quadruples, and so on*

# Monotonicity to the Rescue




Itemsets for items A,B,C,D,E

Neglecting supersets of infrequent pair {A,B}

Adopted from `mmds.org`

# A-Priori Algorithm: Motivation

In the following, we focus on determining frequent pairs.

*N*aive Approach

Consider the algorithm

- For each basket, use double loop to generate all pairs contained in it
- For each pair generated, add 1 to its count
- Store counts using triangular or triples method
- At the end, run through all pairs and determine those whose counts exceed support threshold $s$
- *Benefit:* Only one pass through all baskets
- *Issue:* Number of pairs considered usually does not fit in main memory

# A-Priori Algorithm: Motivation

In the following, we focus on determining frequent pairs.

*N*aive Approach

- *Possible Benefit:* Only pass through all baskets
- *Issue:* Number of pairs considered usually does not fit in main memory

*S*olution: A-Priori-Algorithm

- Have *two passes through baskets* instead of one
- In first run, determine candidate pairs, for which counts are stored
- In second run, determine counts for candidate pairs
- Finally filter for frequent pairs

# A-Priori Algorithm: First Pass

*Create and Maintain Two Tables*

- *First table A:* Let $x$ be an item name, then $A[x]$ reflects that $x$ is the $A[x]$-th item in the order of their appearance in the basket file
- *Second table B:* Let $k$ be an item number, then $B[k]$ is the number of baskets in which item number $k$ appears

*Read Baskets: Fill Table B*

- For each basket, for each item $x$ in the basket, do

$$B[A[x]] = B[A[x]] + 1 \tag{13}$$

- That is, iteratively increase item counts while running through all items in all baskets

# A-PRIORI ALGORITHM: SECOND PASS I

- Let $n$ be the number of items
- Let $m$ be the number of items found to be *frequent*
- By user constraints, usually $m << n$

*Create Third Table*

- *Third table C:* Let $1 \leq k \leq n$ be an item number. Then

$$C[k] = \begin{cases} 0 & \text{if item number } k \text{ is not frequent} \\ l & \text{if item number } k \text{ was found the } l\text{-th frequent item} \end{cases} \quad (14)$$

So, $C \in \{0, 1, ..., m\}^n$, where

- $C[k] = 0$ $n - m$ times
- $C[k] = i, 1 \leq i \leq m$ exactly one time
- $0 < C[k_1] < C[k_2]$ implies $k_1 < k_2$, expressing that $C$ preserves the order of appearance of items

# A-Priori Algorithm: Second Pass II

*Count Pairs Data Structure*

- Use either triangular or triples method data structure to hold counts
  - For using triangular method, renumbering necessary
- By monotonicity, a pair can only be frequent, if both items are frequent
- So, space required is $O(m^2)$ rather than $O(n^2)$
  ☞ $m << n$ implies $m^2 << n^2$, so fits in main memory!

*Examine Baskets*

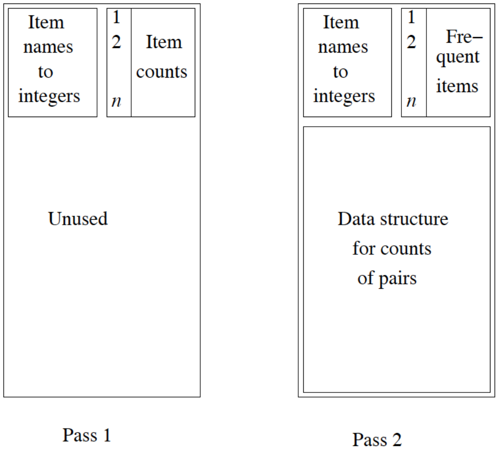
1. For each basket, for each item $x$, see whether

$$C[A[x]] > 0 \quad \text{that is, whether } x \text{ is frequent} \tag{15}$$

2. Using double loop, generate all pairs of frequent items in the basket
3. For each such pair, increase count by one in pair count data structure

*Eventually:* examine which pairs are frequent in pair count data structure

# A-Priori Algorithm: Main Memory Usage



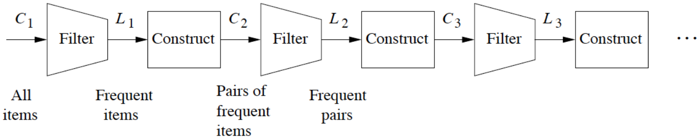

Use of main memory during A-Priori passes

Adopted from `mmds.org`

# A-Priori Algorithm: All Frequent Itemsets

- *One extra pass* for each $k > 2$ to mine frequent itemsets of size $k$
- The A-Priori algorithm proceeds iteratively
  - Mining frequent itemsets of size $k + 1$ is based on knowing frequent itemsets of size $k$
- Each iteration consists of two steps for each $k$:
  - Generate a candidate set $C_k$
  - Filter $C_k$ to produce $L_k$, the truly frequent itemsets of size $k$
- The algorithm terminates at first $k$ where $L_k$ is empty
  - Monotonicity says we are done mining frequent itemsets

# A-Priori Algorithm: Candidate Generation and Filtering




A-Priori algorithm: Alternating between candidate generation and filtering

Adopted from `mmds.org`

- *Construct:* Let $C_k$ be all itemsets of size $k$, every $k-1$ of which belong to $L_{k-1}$
- *Filter:* Make a *pass through baskets* to count members of $C_k$; those with count exceeding $s$ will be part of $L_k$
    - For storing counts for itemsets of size $k$, extend triples method
    - E.g. storing quadruples for frequent triples, and so on...

# Materials / Outlook

- See *Mining of Massive Datasets*, sections 5.5, 6.1, 6.2
- As usual, see `http://www.mmds.org/` in general for further resources
- Next lecture: 'Frequent Itemsets II / Recommendation Systems"
  - See *Mining of Massive Datasets*, 6.3, 6.4.5, 9.1, 9.2